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A generalized theory of fracture mechanics

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A fracture-mechanics theory is developed which gives fracture criteria for solids in general, without limitations as to their linearity, elastic behaviour or infinitesimal strain. Besides the "standard" results of the theory which reduce to familiar forms like the Griffith equation for linear, elastic solids, several new results emerge from the theory. These include a relationship between the surface work and the true surface energy of the solid, an explanation of certain departures from standard fracture mechanics obtained with inelastic materials, and a prediction and explanation of the phenomenon of notch brittleness. Further applications of the theory, such as adhesive failure and fatigue, will be explored in a subsequent paper.

1. Introduction

The concepts of fracture mechanics, originating with Griffith [1] in 1920 and extended by Orowan [2], Irwin [3], Benbow and Roessler [4], Rivlin and Thomas [5] and others [6], are fundamental to the understanding of fracture processes and to their characterization. Fracture mechanics enable us to distinguish the intrinsic resistance to fracture of elastic solids (e.g. their fracture toughness) from geometrical factors (such as flaw size) which also affect such parameters as tensile strength, fracture strain and so on. The determination of fracture toughness is now a standard procedure in materials testing and this parameter is increasingly employed in engineering specifications.

The growing importance of fracture mechanics and of parameters derived therefrom underline, however, some serious deficiencies in the theory. These can be enumerated as follows.

1. Linear fracture mechanics is based upon the assumptions of linear elasticity and infinitesimal strain. Both of these assumptions break down in the highly strained vicinity of the crack tip, especially when plastic zones are formed. The very high values typically observed for "surface energy" using the Griffith approach bear eloquent testimony to the inelastic nature of the deformations. Yet these values are derived on the assumption of elastic behaviour.

2. The elastic equations commonly used to characterize the stress distribution around a crack are only approximate even if the material is everywhere elastic e.g. they require zero tip radius and fail to predict the correct stress at large distances from the crack tip.

3. Even the interesting and elegant work of Rice and others [13-15] on strain fields in elastic-plastic and strain-hardening solids (see later) is really a non-linear elastic treatment and does not take into account the energy losses which manifest themselves as the crack begins to propagate.

4. The energy balance approach developed by Rivlin and Thomas [5] for rubber, whilst it avoids the restriction of infinitesimal strains, still requires the material to be reversibly elastic (as does linear fracture mechanics). Needless to say, real materials are not reversible, especially when the strains are large.

Many attempts have, of course, been made to compensate for these effects. For example, crack lengths are "corrected" by adding the length of any plastic zone which may form, and "constants" in the Rivlin-Thomas approach are allowed to vary to cope with the deficiencies of theory. Such devices, however, not only reduce the elegance of the theory but are soon exhausted as attempts are made to extend fracture mechanics treatments to a wider range of materials and phenomena. In some cases fracture mechanics has to be abandoned in favour of empirical criteria for fracture such as critical crack-opening displacement [7]. Even where the fracture mechanics approach works satisfactorily its theoretical foundation is often decidedly insecure.

This paper is an attempt to encourage a new look at fracture mechanics theory. In particular it examines a generalized approach which removes the serious constraints imposed by the three requirements of linear elasticity, reversibility and infinitesimal strains, with a great gain in generality at only a modest sacrifice of detail. Using this method several "classical" results are derived and thus shown to be far more general than previous considerations would imply. Some new results are also derived which illuminate the physical mechanisms of fracture and provide rather more insight into the nature of fracture "surface energies" than exists at present.

The new approach still requires much work, but it is hoped that these initial considerations will stimulate the interest and activity of others in this field. Certain aspects of the new theory have already been published in the context of experimental studies on fatigue [8] in polyethylene and the adhesion of elastomers [9, 10]. Because of the much wider implications of the theory, however, it has been felt appropriate to publish this present more general and extended treatment.

2. Propagation of a crack in an infinite elastic lamina

We begin by considering the case of an infinite sheet of material containing a crack of length $2c$ and loaded at infinity by a uniform stress applied at right angles to the crack axis. Initially the material is considered to be elastic and this constraint is relaxed in Section 6 of the paper. It is convenient to assume a zero tip radius for the crack, but no other assumptions are required. In particular no requirements are made concerning linearity or the magnitude of the strains. The situation is sketched in Fig. 1, where X, Y are cartesian co-ordinates (in the plane of the lamina) of the point P, referred to a fixed origin at the centre of the crack and to the undeformed state.

Let the applied stress at infinity be σ_0 and the corresponding energy density in the material at infinity be W_0 and let these be held constant during propagation of the crack. (Note that W_0 will only be stored energy density for an elastic lamina; for the inelastic case, to be treated later,

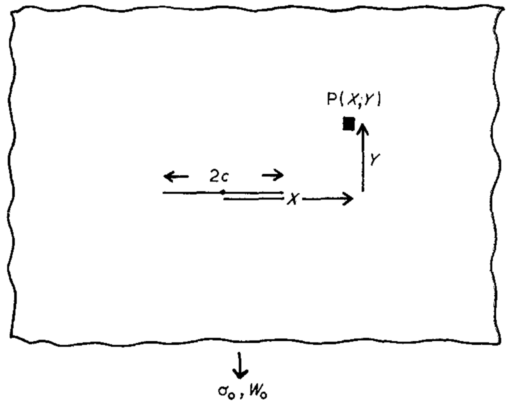

Figure 1 Infinite lamina containing a crack.

W_0 is the work done on unit volume of material during a monotonic deformation i.e. the "input energy density". The same applies to the local energy density W at any point in the stress field.)

From dimensional considerations only,

$$\left.\begin{aligned} \sigma_{ij}(\mathrm{P}) &= \sigma_0 f_{ij}\left(\frac{X}{c}, \frac{Y}{c}, \epsilon_0\right) \quad &(1a) \\ W(\mathrm{P}) &= W_0 f\left(\frac{X}{c}, \frac{Y}{c}, \epsilon_0\right) \quad &(1b) \end{aligned}\right\}$$

where $\sigma_{ij}(\mathrm{P})$ is a component of the stress tensor at P, and f is a function.

The strain at infinity, ϵ_0, serves to characterize the overall level of constraint in the system. For linear materials the inclusion of ϵ_0 is redundant, but for non-linear solids the spatial distribution functions f are not independent of the local stress or strain levels. This dependence is allowed for by inclusion of ϵ_0 since the local strain tensor at P is uniquely defined by ϵ_0 and the spatial variables x, y (see below).

We introduce reduced variables,

$$x = X/c \; ; \; y = Y/c \tag{2}$$

so that,

$$W(\mathrm{P}) = W_0 f(x, y, \epsilon_0) \, . \tag{3}$$

The change in energy density at P due to an increment of crack growth is, in the limit, at constant W_0, ϵ_0,

$$\begin{aligned} \frac{\mathrm{d}W(\mathrm{P})}{\mathrm{d}c} &= W_0\left[\left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial c}\right) + \left(\frac{\partial f}{\partial y}\frac{\partial y}{\partial c}\right)\right] \\ &= -\frac{W_0}{c}\left[\left(x\frac{\partial f}{\partial x}\right) + \left(y\frac{\partial f}{\partial y}\right)\right] \\ &= -\frac{W_0}{c}\, g(x, y, \epsilon_0) \end{aligned} \tag{4}$$

where g is another function.

The total energy change in the system due to propagation of the crack is

$$\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}c} = \sum_{\mathrm{P}} \frac{\mathrm{d}W(\mathrm{P})}{\mathrm{d}c}\,\delta v \tag{5}$$

where $\mathscr{E}$ is the total energy in the system and δv is the volume element at P, that is

$$\begin{aligned}\delta v &= h\delta X\,\delta Y\\ &= hc^2\,\delta x\delta y\end{aligned} \tag{6}$$

where h is the undeformed lamina thickness. Thus, from Equations 4-6

$$\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}c} = -\,W_0ch\sum_{\mathrm{P}} g(x, y, \epsilon_0)\delta x\delta y\,\cdot \tag{7}$$

Referring the energy change to unit area of crack interface A and remembering that $A = 4\,ch$,

$$\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}A} = \frac{-\,W_0c}{4}\sum_{\mathrm{P}} g(x, y, \epsilon_0)\delta x\delta y\,. \tag{8}$$

Now the summation involves only ϵ_0 and geometrical terms and is carried out in dimensionless (i.e. x, y) space where the reduced crack length is always unity. The summation is thus independent of crack length. This still applies if the boundary of integration is defined by some such condition as $\sigma(\mathrm{P}) = \sigma_0$ since this boundary, from Equation 1 is itself determined by x, y. We may thus write,

$$-\,\frac{d\mathscr{E}}{\mathrm{d}A} = k_1\,(\epsilon_0)\,cW_0 \tag{9}$$

where k_1 is a function of ϵ_0 only. The LHS is of course the energy *available* per area of crack interface to propagate the crack. The RHS reduces immediately to Griffith's formula for a linear material for which, of course,

$$W_0 = \sigma_0^2/2E$$

provided k_1 is replaced by π,

$$-\,\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}A} = \frac{\pi c\sigma_0^2}{2E}\,. \tag{10}$$

Notice, however, that Equation 9 is valid for *any* elastic material regardless of linearity or strain magnitude, and thus has far greater generality than the Griffith equation.

When crack propagation occurs, the energy *available* (Equation 9) equals the energy, $\mathscr{T}$, *required* to create unit area of new interface, i.e.

$$k_1\,(\epsilon_0)\,cW_0 = \mathscr{T} \tag{11}$$

where, of course, $\mathscr{T}$ is the usual "fracture energy" or "surface work" and is commonly regarded as a characteristic of the material.

The dependence of k_1 upon ϵ_0 in Equation 9 is well documented for rubber, where k_1 decreases from π to about 2.0 as ϵ_0 increases from zero to very high values [16].

3. Crack inclined at an angle to the *X*-axis

If the crack be inclined at an angle to the line of action of the applied stress, Equation 1b becomes,

$$W(\mathrm{P}) = W_0 f\left(\frac{X}{c}, \frac{Y}{c}, \theta, \epsilon_0\right) \tag{12}$$

where θ is the angle the crack axis makes with the X-axis. On differentiating,

$$\frac{\mathrm{d}W(\mathrm{P})}{\mathrm{d}c} = \frac{W_0}{c}\left[\left(x\,\frac{\partial f}{\partial x}\right) + \left(y\,\frac{\partial f}{\partial y}\right) + c\left(\frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial c}\right)\right]$$

in which $\partial\theta/\partial c$ is zero if the crack does not change direction.

Thus Equation 4 becomes

$$\frac{\mathrm{d}W(\mathrm{P})}{\mathrm{d}c} = \frac{-\,W_0}{c}\,g(x, y, \theta, \epsilon_0)$$

and finally,

$$-\,\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}A} = k_1(\theta, \epsilon_0)cW_0\,\cdot \tag{13}$$

The effect of inclining the crack is thus to modify only the term k_1, which will from general considerations be a maximum when $\theta = 0$ and zero when $\theta = \pi/2$. However, this only applies if the crack does not change direction.

4. The pure-shear specimen of Rivlin and Thomas

The analysis given above can be applied to a variety of different geometrical arrangements. To illustrate this we take the case of an infinitely long lamina containing a crack of semi-infinite length, constrained between infinite parallel grips separated by a distance l in the undeformed state (see Fig. 2). This case corresponds to the "pure shear" specimen by Rivlin and Thomas and so named because the state of stress set up in the lamina when the grips are separated is one of pure shear (except in that half of the specimen severed by the crack and regions close to the crack tip).

This case is a second example in which only one linear dimension in the X-Y plane is neither

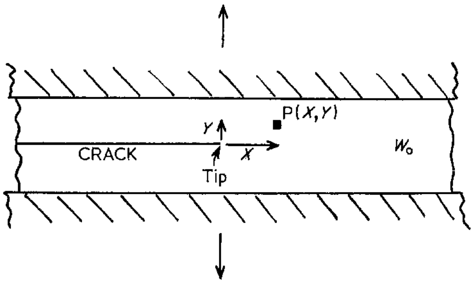

Figure 2 The "pure shear" test specimen.

zero nor infinite and is thus "available" for reduction of variables. In this case the reduced variables are,

$$x = X/l;\ y = Y/l \qquad (14)$$

and the energy density at infinity, W_0, is the energy density in pure shear in unperturbed regions of the lamina remote from the crack.

The only identifiable origin for our co-ordinate system in this case is the crack tip, so that the origin moves with the advancing crack. Then, as before,

$$\frac{\mathrm{d}W(\mathrm{P})}{\mathrm{d}c} = W_0\left[\left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial c}\right) + \left(\frac{\partial f}{\partial y}\frac{\partial y}{\partial c}\right)\right].$$

But,

$$\left.\begin{aligned}\partial x/\partial c &= (1/l)\,\partial X/\partial c = -\,1/l\\ \partial y/\partial c &= (1/l)\,\partial Y/\partial c = 0\end{aligned}\right\} \qquad (15)$$

so that,

$$\frac{\mathrm{d}W(\mathrm{P})}{\mathrm{d}c} = \frac{-\,W_0}{l}\frac{\partial f}{\partial x}.$$

Taking the summation as before, we obtain,

$$-\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}A} = \tfrac{1}{2}W_0 l \sum_{\mathrm{P}} \frac{\partial f}{\partial x}\,\delta x\,\delta y \qquad (16)$$

or, letting $\delta x \to 0$, and introducing the limits of $Y (y = \pm \frac{1}{2})$

$$-\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}A} = \tfrac{1}{2}W_0 l \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{d}y \int_{-\frac{1}{2}}^{\infty} \frac{\partial f}{\partial x}\,\mathrm{d}x$$

$$-\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}A} = \tfrac{1}{2}W_0 l\,. \qquad (17)$$

since by Equation 1, $f = 1$ when $W(\mathrm{P}) = W_0$ i.e. at $x = \infty$ and $f = 0$ when $x = -\infty$, i.e. in the severed region of the specimen. Equation 17 agrees with the result obtained by Rivlin and Thomas by elementary reasoning.

5. Cases involving more than one "available" dimension

The cases considered so far were so contrived that all but one linear dimension of the specimen (in the plane of the lamina) were either zero or infinite. Such zero of infinite dimensions will be referred to as "unavailable" for the purpose of reduction of variables. In practice of course, any real test specimen has several available dimensions and the question arises as to how they affect the energy available for crack propagation.

Consider the centre-crack case treated in Section 1 but now allow the specimen to have boundaries at $X = \pm\, a$ and $Y = +\, b$. Then, leaving implicit the dependence of f upon ϵ_0 for brevity,

$$\left.\begin{aligned}W(\mathrm{P}) &= W_0 f\left(\frac{X}{c}, \frac{Y}{c}, \frac{X}{a}, \frac{Y}{a}, \frac{X}{b}, \frac{Y}{b}\right)\\ &\equiv W_0 f(x_1, y_1, x_2, y_2, x_3, y_3)\end{aligned}\right\}. \qquad (18$$

Then

$$\frac{\mathrm{d}W(\mathrm{P})}{\mathrm{d}c} = -\frac{W_0}{c}\left[\left(x_1\frac{\partial f}{\partial x_1}\right) + \left(\frac{c}{a}\frac{\partial X}{\partial c}\frac{\partial f}{\partial x_2}\right) + \left(\frac{c}{b}\frac{\partial X}{\partial c}\frac{\partial f}{\partial x_1}\right) + - - -\right].$$

But $\partial X/\partial c = \partial Y/\partial c = 0$, for a fixed origin, so that,

$$\frac{\mathrm{d}W(\mathrm{P})}{\mathrm{d}c} = -\frac{W_0}{c}\left[\left(x_1\frac{\partial f}{\partial x_1}\right) + \left(y_1\frac{\partial f}{\partial y_1}\right)\right].$$

Of course, f itself is still dependent upon a, b so that we may write,

$$\frac{\mathrm{d}W(\mathrm{P})}{\mathrm{d}c} = -\frac{W_0}{c}\,g(x_1, y_1, c/a, c/b) \qquad (19)$$

since $x_2 = x_1 c/a$ etc. The terms c/a, c/b are constant with respect to summation over x_1, y_1 space so that the energy available for crack propagation becomes finally,

$$-\,\partial\mathscr{E}/\partial A = k_1(c/a, c/b, \epsilon_0)\,cW_0 \qquad (20)$$

which is to be contrasted and compared with Equation 9 from which it differs in that k_1 is now a function of ratios involving the specimen dimensions. Clearly the treatment applies to any number of "available" linear parameters including, for example, the crack tip radius. As c/a, $c/b \to 0$, k_1 reverts to its value for the infinite lamina.

The general form of Equation 20 must embrace all genuine finite-width and similar corrections applied to the infinite lamina formula when using actual test-pieces.

6. Infinite inelastic lamina

We now turn to reconsider the case treated in Section 1 but relaxing the final constraint of classical elasticity, namely the requirement of perfectly elastic (i.e. thermodynamically reversible) deformation. No previous *analysis* of fracture mechanics has done this although, of course, plastic zones have been "allowed for" by various devices, and non-linear *elastic* deformations have been employed to mimic irreversible deformations. The present treatment takes into consideration inelasticity occurring *anywhere* in the specimen and not just around the crack tip. As will be seen, the treatment leads to a general relationship between the "apparent surface energy" or surface work $\mathscr{T}$ expended in the formation of unit area of crack interface, and the true surface energy $\mathscr{T}_0$ defined as the energy required to break unit area of atomic bonds across the fracture plane.

As before we have,

$$W(\mathrm{P}) = W_0 f(x, y, \epsilon_0)\,.$$

As the crack propagates the spatial distribution of energy density changes, but provided the function f is everywhere single valued this change is fully accounted for by differentiation as employed previously (Equation 4). Consider now, however, an inelastic material which has an unloading stress-strain curve which differs from its loading stress-strain curve (Fig. 3). The energy density is no longer a single valued function of stress or strain so that, in general, it will no longer be a single valued function of the spatial variables x, y. Thus, whilst for elements subject to monotonic loading we may still write, leaving the dependence $f = f(\epsilon_0)$ implicit for brevity,

$$W(\mathrm{P}) = W_0 f(x, y)$$

we must allow a different function for elements undergoing *unloading* from a stress tensor σ_P,

$$W(\mathrm{P}) = W_0\, F(x, y, \sigma_\mathrm{P}) \qquad (21)$$

where

$$(F)_{\sigma=0} = f \qquad (22)$$

but otherwise $F \neq f$. The dependence on σ_P must be allowed because the unloading stress-strain curve is not unique but depends on the maximum stress levels achieved in the stress cycle. Clearly Equation 22 applies because as stress tends to zero all materials ultimately behave elastically.

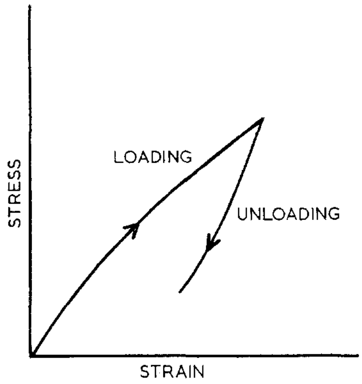

Figure 3 Inelastic deformation.

We thus have,

$$\frac{\mathrm{d}W(\mathrm{P})}{\mathrm{d}c} = \begin{cases} -\dfrac{W_0}{c}\left\{g(x, y)\right\} & \text{for elements loading as the crack propagates} \\[2ex] -\dfrac{W_0}{c}\left\{G(x, y, \sigma_\mathrm{P})\right\} & \text{for elements unloading} \end{cases}$$

where the function G is to F as g is to f. Summing over the stress field gives

$$-\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}A} = cW_0\left[\left(\sum_{\mathrm{P_L}} g(x, y)\,\delta x\,\delta y\right) + \left(\sum_{\mathrm{P_U}} G(x, y, \sigma_\mathrm{P})\,\delta x\,\delta y\right)\right] \qquad (23)$$

where L, U stand for loading, unloading respectively. Equation 23 can be put into a more helpful form by reference to Fig. 4. This is a schematic plot of energy density increments $\pm\Delta W$ against stress increments $\pm\Delta\sigma_{ij}$, representing perturbations produced in the stress and energy states $\sigma(\mathrm{P})$, $W(\mathrm{P})$ (achieved at the point P by monotonic loading) by incremental growth of the crack. Elastic behaviour is represented by the solid line of slope, S, at the origin which gives

$$\pm\Delta W = S(\pm\Delta\sigma_{ij})\,. \qquad (24)$$

Inelastic behaviour is represented by the broken line which gives

$$\left.\begin{aligned} +\Delta W &= +\,S\Delta\sigma_{ij} \\ -\Delta W &= -\,\alpha S\Delta\sigma_{ij} \end{aligned}\right\} \qquad (25)$$

where $\alpha < 1$, and is the energy density recovered at the point P divided by that which would have been recovered from an elastic solid over the same negative stress increment. (It is assumed

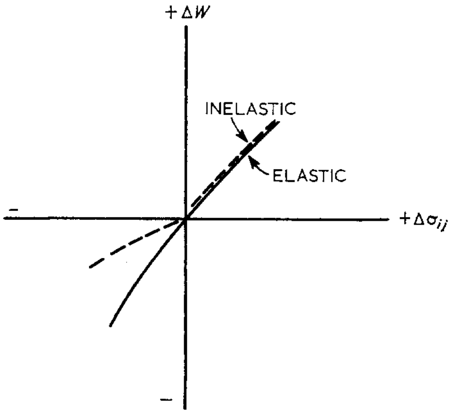

Figure 4 Elastic and inelastic energy density increments.

that no energy is radiated as stress waves.) This enables us to write,

$$\left.\frac{\mathrm{d}W(\mathrm{P})}{\mathrm{d}c}\right|_{\mathrm{U}} = \alpha \left.\frac{\mathrm{d}W(\mathrm{P})}{\mathrm{d}c}\right|_{\mathrm{L}} \cdot \tag{26}$$

The parameter α is, of course, a function of the state of stress at P before crack propagation and of other factors affecting energy loss, namely the temperature T and the rate of strain denoted R.

$$\alpha = \alpha(\sigma_{\mathrm{P}}, T, R)\,. \tag{27}$$

This enables us to re-write Equation 23 as

$$-\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}A} = cW_0 \left[\left(\sum_{\mathrm{P_L}} g(x, y)\, \delta y\, \delta y\right) + \left(\sum_{\mathrm{P_U}} \alpha g(x, y)\, \delta x\, \delta y\right)\right] \tag{28}$$

$$= cW_0 \left[k_1 - \sum_{\mathrm{P_U}} (1 - \alpha)\, g(x, y)\, \delta x\, \delta y\right)\Big] \cdot \tag{29}$$

Let $(1 - \alpha) = \beta$. Since the stress tensor σ_{P} is given by Equation 1 we may write

$$\beta = \beta_0(\sigma_0, T, R)\, g_1(x, y) \tag{30}$$

where g_1 is another function, and

$$-\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}A} = cW_0 \left[k_1 - \left(\beta_0 \sum_{\mathrm{P_U}} g_2(x, y)\, \delta x\, \delta y\right)\right] \tag{31}$$

$$\equiv k_2(\sigma_0, T, R) cW_0\,. \tag{32}$$

The quantity $-\mathrm{d}\mathscr{E}/\mathrm{d}A$ is here the *actual* energy available for forming crack surface *after* deduction of all energy losses throughout the specimen. At the moment of propagation it must, therefore, equal the actual energy requirement per area of interface i.e. the surface energy $\mathscr{T}_0$.

7. Relation between actual and apparent surface energies in fracture

To distinguish between the quantities $-\mathrm{d}\mathscr{E}/\mathrm{d}A$ in Equations 9 and 32 respectively we re-write these equations, re-introducing the ϵ_0 dependence explicitly

$$-\left.\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}A}\right|_1 = k_1(\epsilon_0)\, cW_0 \tag{33}$$

$$= \left.\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}A}\right|_2 = k_2(\sigma_0, T, R, \epsilon_0)\, cW_0 \cdot \tag{34}$$

The first expression is the energy available for crack propagation in a perfectly elastic solid and, when the crack actually propagates, must therefore also equal, $\mathscr{T}$, the *total energy expended by* the system to cause unit area of growth. This is true whether the material is elastic or inelastic. The difference between the two cases is that for the elastic case all this energy is available for bond rupture at the fracture plane whilst in the inelastic case some of this energy is dissipated in the material by inelastic deformation process leaving only a proportion for bond fracture. This remaining proportion is given by Equation 34 and must attain a particular value, $\mathscr{T}_0$, to break the bonds and cause propagation. Under conditions of crack propagation, then,

$$\mathscr{T} = k_1 cW_0 \tag{35}$$

$$\mathscr{T}_0 = k_2 cW_0 \tag{36}$$

whence,

$$\mathscr{T} = \mathscr{T}_0\, k_1/k_2(\sigma_0, T, R) \equiv \mathscr{T}_0 \Phi(\sigma_0, T, R)\,. \tag{37}$$

This important result tells us that the surface work in any material is given by the energy necessary to break unit area of bonds across the fracture plane (i.e. the surface energy according to one definition) multiplied by a "loss function" Φ which varies with the external constraint σ_0 (or W_0), the temperature and the rate of crack propagation, and any other factor affecting the loss characteristics of the material. Φ reduces to unity if the material is everywhere perfectly elastic.

8. Some observations on the behaviour of the loss function Φ

We have,

$$\Phi = k_1 \Big/ \left[k_1 - \left(\beta_0 \sum_{\mathrm{P_U}} g_2(x, y)\, \delta x\, \delta y\right)\right] \cdot \tag{38}$$

For zero energy loss ($\beta_0 = 0$) and $\Phi \rightarrow 1$, but conversely as β_0 increases it attains some finite value at which $\Phi \rightarrow \infty$. At this juncture, Equation 35 shows crack propagation is no longer possible and wholesale deformation of the body takes place. A good example of this is chewing gum and the good adhesive properties of this material are a direct consequence of the same result! In such materials β_0 is sufficiently high to give infinite values for Φ. The transition from crack propagation to wholesale inelastic deformation is given by

$$k_1 = \beta_0 \sum_{P_U} g_2(x, y)\, \delta x\, \delta y \qquad (39)$$

and is thus independent of crack length. The fact that, in general, β_0 will increase with increasing σ_0, (or W_0) has two results. Firstly it means that Φ and thus $\mathscr{T}$ will increase with increasing σ_0, i.e. the energy requirement for propagation is *not* a material constant for a given rate and temperature, but increases with the stress or strain level in the specimen.

This effect is superimposed upon the dependence of k_1 upon ϵ_0 discussed in Section 2 and arising from non-linearity in an *elastic* solid.

The effect will also have a characteristic manifestation in the plots commonly used to determine $\mathscr{T}$, namely those in which the critical value of W_0 to cause propagation (σ_0^2 in the case of near Hookean solids) is plotted against c^{-1} to give a straight line graph. If $\mathscr{T}$ were independent of W_0 such a straight line would pass through the origin, since,

$$\mathscr{T} c^{-1} = k_1 W_0 \,. \qquad (40)$$

If, however, $\mathscr{T}$ increases with W_0, the plot will be deflected to give an apparent intercept at some positive c^{-1} value. This effect is found in many published data, notably that of Berry [11] for PMMA as quoted along with more recent results, by Reed and Squires [12] (see Fig. 5). Some confusion arises because errors arising from the use of infinite-plate formulae for finite-width specimens produces the same effect, but the effect persists even when finite width corrections are made. No doubt many examples exist for the more ductile metals which may not have been published because of the failure of the data to fit linear fracture mechanics.

The second effect of the Φ (σ_0) dependence is potentially even more striking. For large values of c, the condition for propagation (Equation 35) is satisfied for relatively low values of W_0 at which β_0 and thus Φ have their minimum values.

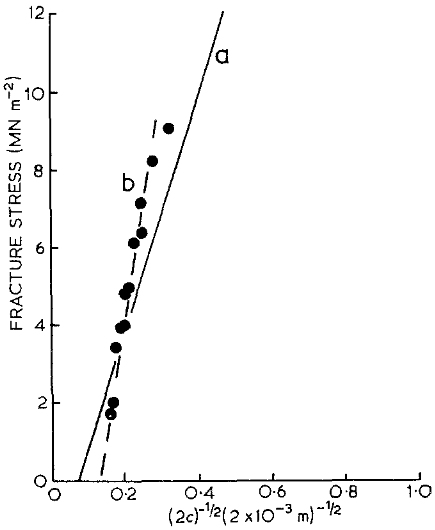

Figure 5 Data from (a) Berry's quasi-static tests and (b) Reed and Squires hypersonic tests on PMMA showing intercepts when fracture stress is plotted against the square root of reciprocal crack length.

It is possible, as c decreases and W_0 increases to maintain the condition for propagation, that β_0 increases with W_0 sufficiently to cause Φ to go to infinity (Equation 38). If this occurs (and this will depend on the magnitudes of β_0 and $\sum_{P_U}$ in Equation 38 as well as the dependence of β_0 on W_0), the phenomenon of *notch brittleness* is predicted. That is, crack propagation is possible only for c greater than some critical value. For lower c values, $\Phi \rightarrow \infty$ and wholesale inelastic deformation of the body will occur rather than crack propagation.

9. Conclusion

It is clear from Section 8 that the relatively explicit form of Φ, along with Equation 37 of Section 7 provide a powerful tool for the understanding and even prediction of fracture phenomena in an unlimited class of solids. In a subsequent publication the power of the new approach will be illustrated by application to phenomena such as adhesive failure and fatigue.

It has not yet been possible to explore the relation, if any, between the new theory and the work of Rice [13] and others [14, 15] on the contour integral method for solving stress and strain distributions around crack tips. This also must be postponed to a subsequent paper. However, certain points can be made even at this stage. Firstly, the present theory, like that of Rice, concentrates attention on energy density as a fundamental characteristic of the "stress" field. Secondly, the contour integral method permits the consideration of non-linear materials, as does the present work. However, a fundamental difference arises from the fact that Rice's approach considers only the *loading* of elements i.e. in effect it considers a non-linear *elastic* [14] situation. The present theory still seems to be the only one in which the *unloading* of inelastic elements is taken into account. It is difficult to see how any fracture criterion can be valid which neglects this aspect.

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